BUSSTEPP Homework 3:
More Exercises on Numerical Quantum Mechanics
Other Exercises

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1 Numerical Exercises

1.1 Harmonic Oscillator

We continue with the harmonic oscillator,

$$V(x) = \tfrac{1}{2}m\omega^2 x^2, \tag{1}$$

$$S = ma\left\{\sum_{i=0}^{N-1} \tfrac{1}{2}[x_{i+1} - x_i)/a]^2 + \tfrac{1}{2}(\omega a)^2(x_i/a)^2\right\}. \tag{2}$$

Exercise III.1: Run with the parameters in Table 1 to vary the lattice spacing. Plot E_1, E_2, and E_2/E_1 vs. a. Verify also that the energies do not depend on m. Results are in Fig. 1.

Explain the striking constancy of $E_2/E_1 = 2$.

1.2 Anharmonic Oscillator

Now add an anharmonic term

$$V(x) = \tfrac{1}{2}m\omega^2 x^2 + \lambda x^4, \tag{3}$$

$$S = ma\left\{\sum_{i=0}^{N-1} \tfrac{1}{2}[x_{i+1} - x_i)/a]^2 + \tfrac{1}{2}(\omega a)^2(x_i/a)^2 + (\lambda a^5/ma)(x_i/a)^4\right\}. \tag{4}$$

From first-order perturbation theory

$$E_n = n\omega\left(1 + \frac{3\lambda(n+1)}{2m^2\omega^3}\right) \tag{5}$$

so the correction is small if $\lambda \ll m^2\omega^3$. The energies as a function of λ are shown in Fig. 2

Table 1: Parameters for exploring the dependence on a.

ma	0.5	1	1.5	2	3
ωa	0.5	1	1.5	2	3
N	128	64	44	32	22

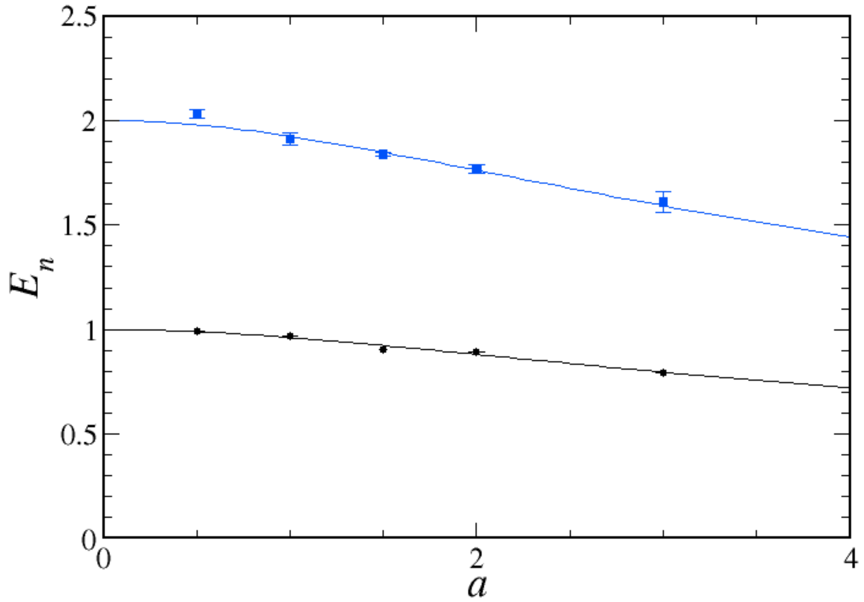

Figure 1: E_1 and E_2 vs. lattice spacing a. The points are Monte Carlo simulation. The lines are the exact solution at non-zero a.

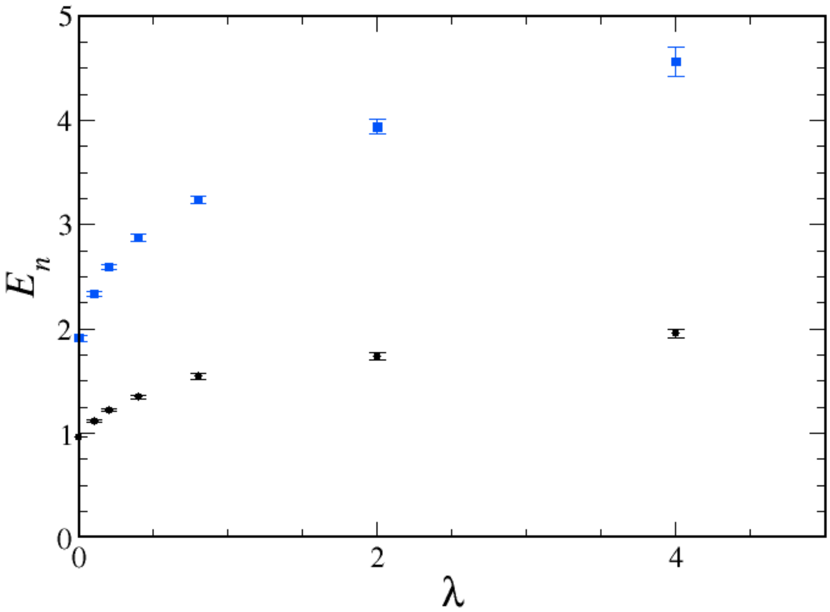

Figure 2: E_1 and E_2 vs. anharmonicity λ, in lattice units.

Exercise III.2: Compute the energies E_1 and E_2 as a function of λ at a lattice spacing so that discretization effects are small. Start with λ small enough so that perturbation theory should be accurate, but extend into the non-perturbative regime.

1.3 Double-well Oscillator

The exercise in this subsection uses the potential

$$V(x) = -\tfrac{1}{2}m\omega^2 x^2 + \lambda x^4 \tag{6}$$

Note the minus sign in front of the quadratic term. There are two minima. Now the first excited state is almost degenerate with the ground state.

Exercise III.3: Return to the program that compute x_{avg} as a function of c. Plot them vs. c. Explain the behavior of x_{avg}, Fig. **??**.

2 Other Exercises

The Standard Yukawa interactions of quarks are

$$\mathcal{L}_Y = -\sum_{i,j=1}^{G}\left[\hat{y}^d_{ij}\bar{Q}^i_L\phi\, D^j_R + \hat{y}^u_{ij}\bar{Q}^i_L\tilde{\phi}\, U^j_R + \text{h.c.}\right], \tag{7}$$

with hypercharges $Y_U = 2/3$, $Y_D = -1/3$, $Y_Q = 1/6$.

Exercise III.4: What must the hypercharge of the Higgs doublet(s) be?

In continuum gauge theories the parallel transporter (or Wilson line) is defined to be

$$U(x,y) = \mathsf{P}\exp\left(\int_y^x dz\cdot A\right). \tag{8}$$

Exercise III.5: Show that $U(x,y) \to g(x)U(x,y)g^{-1}(y)$ under gauge transformations.

The Wilson plaquette action is

$$S = \frac{\beta}{2N}\sum_{x,\mu,\nu} P_{\mu\nu}(x) \tag{9}$$

where

$$P_{\mu\nu} = \operatorname{Re}\operatorname{tr}[1 - U_\mu(x)U_\nu(x+ae^{(\mu)})U^\dagger_\mu(x+ae^{(\nu)})U^\dagger_\nu(x)]. \tag{10}$$

Exercise III.6: Show that the plaquette action reduces to the Yang-Mills action when $a \to 0$.